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Action-angle representation of Multisolitons

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Abstract. For a class of completely integrable, finite dimensional multi-soliton systems the full set of action/angle variables is constructed. The main tools are the well known symmetries and mastersymmetries of the corresponding hamiltonian soliton equation and the spectral properties of the recursion operator. The relationship between these quantities and the action/angle representation is expressed in terms of the asymptotic data provided by the Inverse Scattering Method. As one important interim result we obtain the embedding of the solution manifold of multi-solitons into the complete solution space. Furthermore, we are able to obtain the eigenstates of the recursion operator in an extremely simple way.

INTRODUCTION

In this letter we consider translation invariant hamiltonian evolution equations

$$u_t = K_1(u) \tag{1}$$

in $1+1$-dimensions which admit an infinite number of pairwise commuting symmetries K_n. We restrict ourselves to the case that a localized recursion operator exists. The solution manifold M_N of the stationary equations (for variable $\alpha_0, ..., \alpha_N$)

$$\alpha_0 K_0 + \alpha_1 K_1 + \ldots + \alpha_N K_N = 0$$

is known to be an invariant, finite dimensional subspace of all solutions. In fact, M_N consists of all N-soliton solutions of (1). In case of vanishing boundary conditions at infinity, the flow (1) induces a $2N$-dimensional hamiltonian system on M_N which is completely integrable in the Liouville sense. The Inverse Scattering Method provides the action/angle representation of this integrable equation in terms of the asymptotic velocities and phases (scattering data) of the N-soliton solutions. In this letter we examine the structure of M_N from the geometrical point of view. We give the explicit embedding of M_N into the complete solution space of (1) by determing the tangent bundle of M_N. The main tool for this result are the interrelations between the symmetries K_n and the mastersymmetries τ_n. Then, using the spectral properties of the recursion operator, we are able to give combinations V_i of the symmetries and combinations W_i of the mastersymmetries which are hamiltonian vector fields on M_N. Since these fulfill canonical commutator relations they can be interpreted as vector fields corresponding to action/angle variables. Moreover these vector fields are eigenstates of the recursion operator. Taking into account the asymptotic behaviour of the N-solitons u_N, with vanishing boundary conditions at infinity, we are able to relate the gradients of the asymptotic data with the eigenstates V_i and W_i. Furthermore we obtain a set of eigenstates of the recursion operator as the partial derivatives of the N-soliton solution u_N w.r.t. the asymptotic data.

Apart from the facts mentioned so far our results allow the geometrical interpretation of the mastersymmetries as vector field versions of the angle coordinates. This is well known if the mastersymmetries are hamiltonian vector fields ([8]), i.e. if no localized recursion operator exists ([13]). From this point of view we have shown that even in the case where the mastersymmetries are not hamiltonian one can find suitable combinations of them which turn out to be hamiltonian vector fields on the submanifold M_N.

Since the proofs of all these results are lenghtly and somewhat technically we mostly omitted them in this letter. For details and proofs of our statements we refer to the exhaustive paper [9].

RESULTS

As said before we consider on a suitable manifold M the evolution equation $u_t = K_1(u)$ where $u = u(x,t) \in M$ denotes the field variable and $K_1(u)$ is a translation invariant vector field on M. We are interested in equations which admit a localized hereditary ([4]) recursion operator $\Phi(u)$ with an implectic/symplectic factorization ([5]) or equivalent, a compatible hamiltonian pair ([10],[11])

$$\Phi(u) = \Theta_1(u)\,\Theta_0^{-1}(u) =: \Theta_1(u)\,J(u) \ .$$

The operator Φ generates a hierarchy of pairwise commuting symmetries ([6])

$$K_n(u) := \Phi^n(u)K_0(u) = \Phi^n(u)u_x$$

for the equation (1). If there is a scaling quantity $\tau_0(u)$ with $[\tau_0, K_0] = \varrho K_0$ and with $[\tau_0, \Phi A] = \Phi A + \Phi[\tau_0, A]$ for all vector fields A, then the recursive application of Φ on τ_0 produces a second hierarchy of vector fields, the so-called mastersymmetries $\tau_n = \Phi^n\tau_0$ ([3],[7],[14]). Here $[\ ,\ ]$ denotes the usual commutator between vector fields and ϱ is a scalar. Due to the hereditariness of Φ the following commutator relations hold between the symmetries K_n and the mastersymmetries τ_n

$$[K_n, K_m] = 0 \ , \ [\tau_n, K_m] = (m+\varrho)K_{n+m} \ , \quad [\tau_n, \tau_m] = (m-n)\tau_{n+m} \ . \tag{2}$$

EXAMPLE: Let $u = u(x,t)$ be an element of the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$. With $K_0(u) = u_x$ and $\tau_0(u) = \frac{1}{2}xu_x + u$ and the hereditary recursion operator

$$\Phi(u) = D^2 + 2DuD^{-1} + 2u = (D^3 + 2Du + 2uD)D^{-1} = \Theta_1 J$$

we obtain the hierarchy of the well known Korteweg-deVries equation

$$u_t = u_{xxx} + 6uu_x = K_1(u) = \Phi(u)u_x \ .$$

Here D denotes the differential operator $D = \frac{\partial}{\partial x}$ and D^{-1} its inverse.

It is well known that the equations introduced above admit multi-soliton solutions ([1]). The N-soliton solutions can be described as elements of the following invariant submanifold M_N of M ([12])

$$M_N = \{\ u \mid \text{there exists } \alpha_n \text{ such that } \sum_{n=0}^{N} \alpha_n K_n = 0\} \ . \tag{3}$$

Since for vanishing boundary conditions at infinity this manifold can be parametrized by the velocities and the phases of the N-soliton-solutions, it has the dimension $2N$. For M_N we obtain the following result

THEOREM 1:

(1) For all $r,p \in \mathbb{N}_0$ we have the following representation of the tangent space T_uM_N of M_N at the point u

$$T_uM_N = \text{span } \{K_r, K_{r+1}, \dots, K_{r+N-1}, \tau_p, \tau_{p+1}, \dots, \tau_{p+N-1}\} \ .$$

(2) The commutator relations (2) remain valid on M_N.

(3) Whenever the α_n are the coefficients given by (3) in the definition of the manifold point u then the following hold

(i) For all $r \in \mathbb{N}_0$ we have the following identities on M_N:

$$\sum_{n=0}^{N} \alpha_n K_{n+r} = 0 \quad \text{and} \quad \sum_{n=0}^{N} \alpha_n \tau_{n+r} = 0 \ .$$

(ii) The discrete eigenvalues $c_1, ..., c_N$ of Φ are given as the zeros of the characteristic polynominal $P(\xi) = \sum_{n=0}^{N} \alpha_n \xi^n$.

(iii) The corresponding eigenstates are $\tilde{V}_i = \Pi_i(\Phi) K_0$ and $\tilde{W}_i = \Pi_i(\Phi)\tau_0$, where $\Pi_i(\xi) = P(\xi)/(\xi - c_i)$.

(4) The restriction $\bar{J} := J_{|red}$ of J to M_N induces a non-degenerated symplectic form on M_N.

As a direct consequence of theorem 1 we obtain that the recursion operator Φ leaves the tangent space $T_u M_N$ of the reduced manifold invariant. Hence, the restriction $\bar{\Phi} := \Phi_{|red}$ of Φ to M_N is a linear operator on a finite dimensional space. This operator $\bar{\Phi}$ has the following properties:

THEOREM 2:

(1) $\bar{\Phi}$ is invertible and $\bar{\Phi}\ \bar{J}^{-1}$ is skew-symmetric.

(2) The eigenvalues $c_1, ..., c_N$ of $\bar{\Phi}$ are doubly degenerated.

(3) Renorming of the eigenstates $\tilde{V}_i$,$\tilde{W}_i$ leads to eigenstates V_i and W_i of $\bar{\Phi}$ for the eigenvalue c_i which are hamiltonian vector fields w.r.t. $\bar{J}^{-1}$.

(4) The eigenstates V_i and W_i fulfill the commutator relations

$$[V_i, V_j] = 0 = [W_i, W_j] \quad , \quad [V_i, W_j] = \delta_{ij} \ . \tag{4}$$

Since the eigenstates V_i,W_i are hamiltonian vector fields and since they fulfill the canonical commutator relations (4) ([15]), their potentials can be interpreted as action/angle variables for the flow induced by (1) on M_N.

Up to this stage all our results are of a purely algebraically nature. We now concentrate more intensively on the case of N-soliton solutions with vanishing boundary conditions at infinity, decomposing into N single solitons for $t \to \pm\infty$

$$u_N \cong \sum_{i=1}^{N} s_i(x + c_i t + q_i) \ .$$

In this case we get a simple method for finding the eigenstates of the recursion operator.

THEOREM 3: Taking the partial derivatives of u_N w.r.t. the asymtotic data

$$\frac{\partial u_N}{\partial q_i} \quad \text{and} \quad \frac{\partial u_N}{\partial c_i}$$

one obtains eigenstates of the recursion operator $\bar{\Phi}$ for the eigenvalue c_i.

Of course, this is well known for the derivative w.r.t. the phases q_i ([2]), but it may come as a surprise that the second set of eigenstates is simply given by $\partial u_N/\partial c_i$.
Moreover we are able to express the gradients of the global asymptotic data in terms of the local quantities $\bar{J}V_i$ and $\bar{J}W_i$. The superposition formula for phase shifts can be obtained in full generality. All the explicit formulas are given in [9]. There we also exhibit several examples and we show plots of the x-derivative of the densities of the action and the angle variable. Furthermore a comparison of our approach with the work of other people can be found in that paper.

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